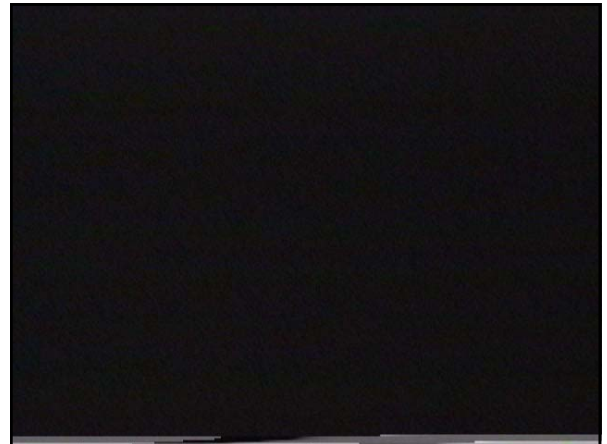


Video Segment

Juggling Robot, Dan Koditschek
University of Michigan,
ISRR'93 video proceedings

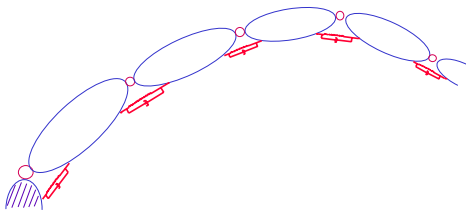


Robot Control

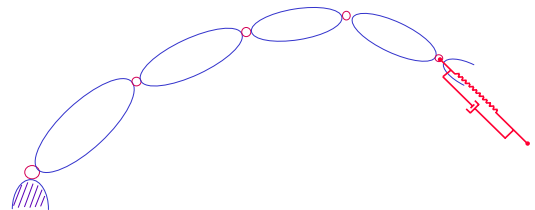
Control

- Natural Systems
- PID Control
- Joint-Space Dynamic Control
- Task-Oriented Control
- Force Control

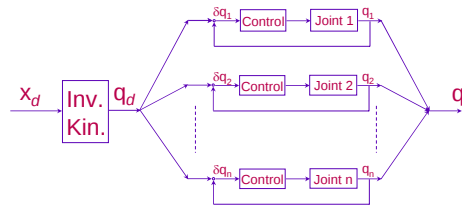
Joint-Space Control



Task-Oriented Control



Joint Space Control



Resolved Motion Rate Control (Whitney 72)

$$\delta x = J(\theta) \delta \theta$$

Outside singularities

$$\delta \theta = J^{-1}(\theta) \delta x$$

Arm at Configuration θ

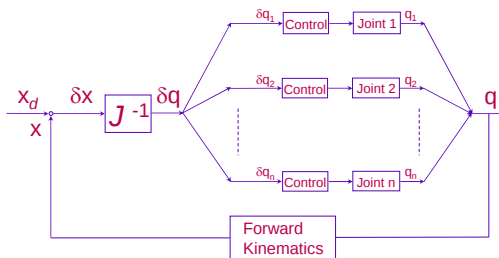
$$x = f(\theta)$$

$$\delta x = x_d - x$$

$$\delta \theta = J^{-1} \delta x$$

$$\theta^+ = \theta + \delta \theta$$

Resolved Motion Rate Control



Natural Systems

Conservative Systems 

$$\frac{d}{dt} \left(\frac{\partial (K-V)}{\partial \dot{x}} \right) - \frac{\partial (K-V)}{\partial x} = 0 \quad K = \frac{1}{2} m \dot{x}^2$$

Natural Systems

Conservative Forces 

$$\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{x}} \right) - \frac{\partial K}{\partial x} = - \frac{\partial V}{\partial x}$$

$$m \ddot{x} = F = -kx$$

Potential Energy of a spring

$$V = \text{Work} = \int_x^0 (-kx) \delta x = \frac{1}{2} kx^2$$

Natural Systems

Conservative Forces 

$$\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{x}} \right) - \frac{\partial K}{\partial x} = - \frac{\partial V}{\partial x}$$

Potential Energy of a spring

$$V = \text{Work} = \int_x^0 (-kx) \delta x = \frac{1}{2} kx^2$$

$$m \ddot{x} = F = -kx$$

$$- \frac{\partial}{\partial x} \left(\frac{1}{2} kx^2 \right)$$

Natural Systems

Conservative Systems



$$\frac{d}{dt} \left(\frac{\partial(K-V)}{\partial \dot{x}} \right) - \frac{\partial(K-V)}{\partial x} = 0$$

$$m \ddot{x} + kx = 0$$

$$K = \frac{1}{2} m \dot{x}^2$$

$$V = \frac{1}{2} kx^2$$

Natural Systems

Conservative Systems

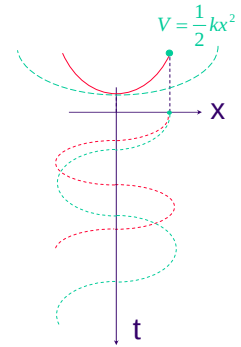
$$m \ddot{x} + kx = 0$$

Frequency increases
with stiffness
and inverse mass

$$\text{Natural Frequency } \omega_n = \sqrt{\frac{k}{m}}$$

$$\ddot{x} + \omega_n^2 x = 0$$

$$x(t) = c \cos(\omega_n t + \phi)$$



Natural Systems

Dissipative Systems



$$\frac{d}{dt} \left(\frac{\partial(K-V)}{\partial \dot{x}} \right) - \frac{\partial(K-V)}{\partial x} = f_{\text{friction}}$$

Viscous friction: $f_{\text{friction}} = -b\dot{x}$

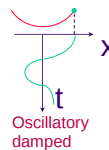
$$m \ddot{x} + b\dot{x} + kx = 0$$

Dissipative Systems

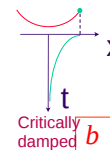


$$m \ddot{x} + b\dot{x} + kx = 0$$

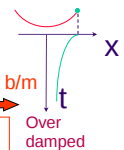
$$\ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = 0$$



Oscillatory
damped



Critically
damped



Over
damped

$$\text{Higher } b/m$$

$$\frac{b}{m} = 2\omega_n$$

2nd order systems

$$m \ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = 0$$

$$\frac{b/m}{2\omega_n} \cdot 2\omega_n$$

Critically
damped
when
 $b/m = 2\omega_n$

Natural damping ratio

$$\xi_n = \frac{b}{2\omega_n m} = \frac{b}{2\sqrt{km}}$$

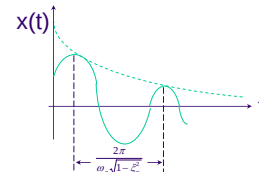
Critically damped system: $\xi_n = 1$ ($b = 2\sqrt{km}$)

Time Response

$$\ddot{x} + 2\xi_n \omega_n \dot{x} + \omega_n^2 x = 0$$

$$\text{Natural frequency } \omega_n = \sqrt{\frac{k}{m}} ; \xi_n = \frac{b}{2\sqrt{km}} \quad \text{Natural damping ratio}$$

$$x(t) = ce^{-\xi_n \omega_n t} \cos(\omega_n \sqrt{1 - \xi_n^2} t + \phi)$$



damped
Natural
frequency

$$\omega = \omega_n \sqrt{1 - \xi_n^2}$$

Example



$$m\ddot{x} + b\dot{x} + kx = 0$$

$$m = 2.0$$

$$b = 4.8$$

$$k = 8.0$$

what is the "damped Natural frequency"

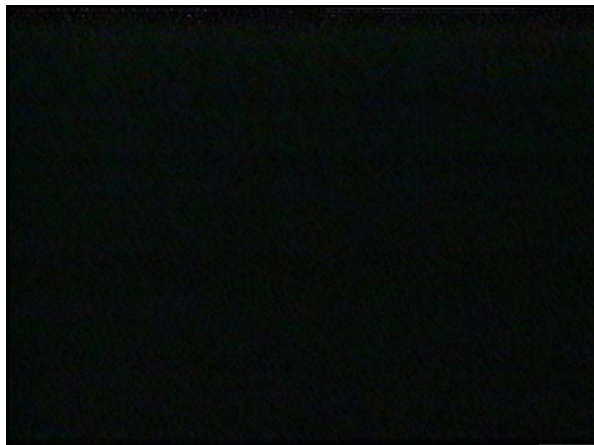
$$\omega = \omega_n \sqrt{1 - \xi_n^2}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 2; \quad \xi_n = \frac{b}{2\sqrt{km}} = 0.6$$

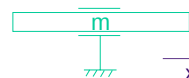
$$\omega = 2\sqrt{1 - 0.36} = 1.6$$

Video Segment

Tactile Sensing, H. Maekawa et al.
MEL, AIST-MITI, Tsukuba, Japan
ICRA'93 video proceedings



1-dof Robot Control



$$m\ddot{x} = f$$

Potential Field

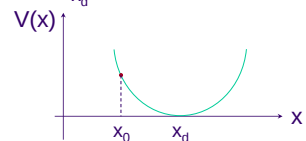
$$V(x) > 0, x \neq x_d$$

$$V(x) = 0, x = x_d$$

$$V(x) = \frac{1}{2} k_p (x - x_d)^2; \quad f = -\nabla V(x) = -\frac{\partial V}{\partial x}$$

$$m\ddot{x} = -\frac{\partial}{\partial x} \left[\frac{1}{2} k_p (x - x_d)^2 \right]; \quad m\ddot{x} + k_p (x - x_d) = 0$$

Position gain



Passive Systems (Stability)

$$V_{goal} = \frac{1}{2} k_p (x - x_g)^T (x - x_g)$$

$$\text{System} \quad \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{x}} \right) - \frac{\partial K}{\partial x} = f$$

$$\Downarrow \quad f = -\frac{\partial V_{goal}}{\partial x}$$

$$\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{x}} \right) - \frac{\partial (K - V_{goal})}{\partial x} = 0 \quad \text{Conservative Forces}$$

Stable

Asymptotic Stability

$$\text{a system} \quad \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{x}} \right) - \frac{\partial (K - V_{goal})}{\partial x} = F_s$$

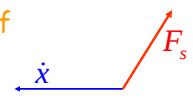
is asymptotically stable if

$$F_s^T \dot{x} < 0; \quad \text{for } \dot{x} \neq 0$$

$$F_s = -k_v \dot{x} \rightarrow k_v > 0$$

Control

$$F = -k_p (x - x_{goal}) - k_v \dot{x}$$



Proportional-Derivative Control (PD)

$$m\ddot{x} = f = -k_p(x - x_d) - k_v\dot{x}$$

$$m\ddot{x} + k_v\dot{x} + k_p(x - x_d) = 0$$

Velocity gain Position gain

$$1. \ddot{x} + \frac{k_v}{m}\dot{x} + \frac{k_p}{m}(x - x_d) = 0$$

$$1. \ddot{x} + 2\xi\omega\dot{x} + \omega^2(x - x_d) = 0$$

$$\xi = \frac{k_v}{2\sqrt{k_p m}} \quad \text{closed loop damping ratio} \quad \omega = \sqrt{\frac{k_p}{m}} \quad \text{closed loop frequency}$$

Gains

$$k_p = m\omega^2$$

$$k_v = m(2\xi\omega)$$

Gain Selection

$$\text{set } \begin{pmatrix} \xi \\ \omega \end{pmatrix} \rightarrow \begin{matrix} k_p = m\omega^2 \\ k_v = m(2\xi\omega) \end{matrix}$$

Unit mass system

$$k'_p = \omega^2$$

$$k'_v = 2\xi\omega$$

m - mass system

$$k_p = m \cdot k'_p$$

$$k_v = m \cdot k'_v$$

Control Partitioning

$$m\ddot{x} = f \implies m(1.\ddot{x}) = m f'$$

$$f = -k_v\dot{x} - k_p(x - x_d)$$

$$f = m[-k'_v\dot{x} - k'_p(x - x_d)] = m f'$$

$$m\ddot{x} = m f' \quad f'$$

$$1. \ddot{x} = f' \quad \text{unit mass system}$$

$$1.\ddot{x} + k'_v\dot{x} + k'_p(x - x_d) = 0$$

$$2\xi\omega \quad \omega^2$$

Non Linearities

$$m\ddot{x} + b(x, \dot{x}) = f$$

Control Partitioning

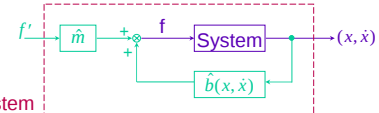
$$f = \alpha f' + \beta$$

$$\text{with } \alpha = \hat{m}$$

$$\beta = \hat{b}(x, \dot{x})$$

$$m\ddot{x} + b(x, \dot{x}) = \hat{m}f' + \hat{b}(x, \dot{x})$$

$$\rightarrow 1. \ddot{x} = f'$$



Unit mass system

Motion Control

$$m\ddot{x} + b(x, \dot{x}) = f \implies \underset{f = mf' + b}{1. \ddot{x} = f'}$$

Goal Position (x_d):

$$\text{Control: } f' = -k'_v\dot{x} - k'_p(x - x_d)$$

$$\text{Closed-loop System: } 1. \ddot{x} + k'_v\dot{x} + k'_p(x - x_d) = 0$$

Trajectory Tracking

$$x_d(t); \dot{x}_d(t); \text{ and } \ddot{x}_d(t)$$

$$\text{Control: } f' = \ddot{x}_d - k'_v(\dot{x} - \dot{x}_d) - k'_p(x - x_d)$$

Closed-loop System:

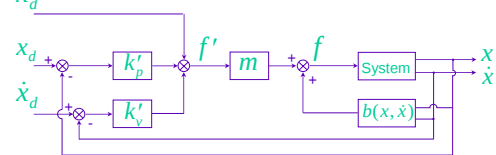
$$(\ddot{x} - \ddot{x}_d) + k'_v(\dot{x} - \dot{x}_d) + k'_p(x - x_d) = 0$$

$$\text{with } e \equiv x - x_d$$

$$\ddot{e} + k'_v\dot{e} + k'_pe = 0$$

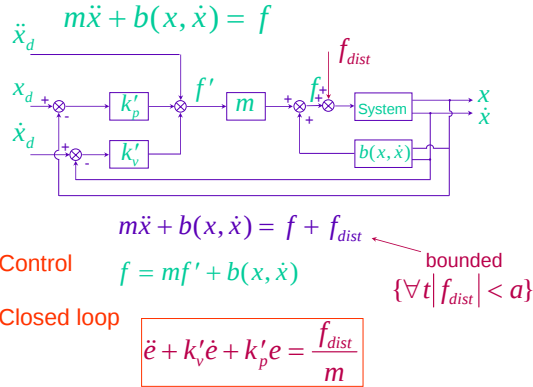
Disturbance Rejection

$$m\ddot{x} + b(x, \dot{x}) = f$$



$$\ddot{e} + k'_v\dot{e} + k'_pe = 0$$

Disturbance Rejection



Steady-State Error

$$\ddot{e} + k'_v \dot{e} + k'_p e = \frac{f_{dist}}{m}$$

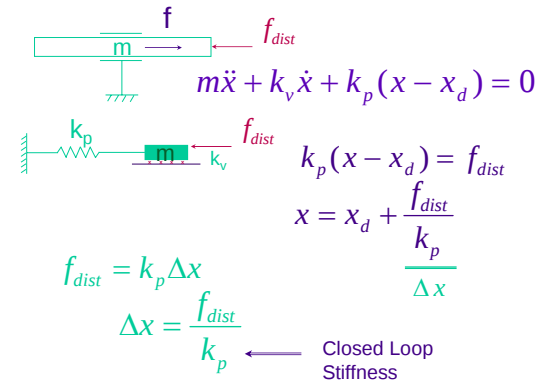
The steady-state ($\dot{e} = \ddot{e} = 0$):

$$k'_p e = \frac{f_{dist}}{m}$$

$$\underline{\underline{e = \frac{f_{dist}}{m k'_p} = \frac{f_{dist}}{k_p}}}$$

Closed loop position gain

Steady-State Error - Example



PID (adding Integral action)

System $m \ddot{x} + b(x, \dot{x}) = f + f_{dist}$

Control $f = m f' + b(x, \dot{x})$

$$f' = \ddot{x}_d - k'_v (\dot{x} - \dot{x}_d) - k'_p (x - x_d) - k'_i \int (x - x_d) dt$$

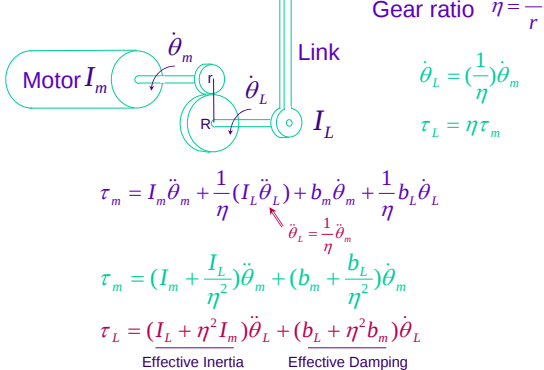
Closed-loop System

$$\ddot{e} + k'_v \dot{e} + k'_p e + k'_i \int e dt = \frac{f_{dist}}{m}$$

constant

Steady-state Error $\underline{\underline{e = 0}}$

Gear Reduction



Effective Inertia

$$I_{eff} = I_L + \eta^2 I_m$$

for a manipulator
 $I_L = I_L(q)$

$\eta = 1$
Direct Drive

Gain Selection

$$k_p = (I_L + \eta^2 I_m) k'_p$$

$$k_v = (I_L + \eta^2 I_m) k'_v$$

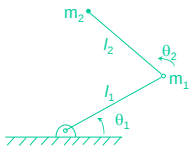
Time Optimal Selection

$$\hat{I}_L = \frac{1}{4} (\sqrt{I_{L_{min}}} + \sqrt{I_{L_{max}}})^2$$

Video Segment

On the Run, Marc Raibert, MIT
ISRR'93 video proceedings

Manipulator Control

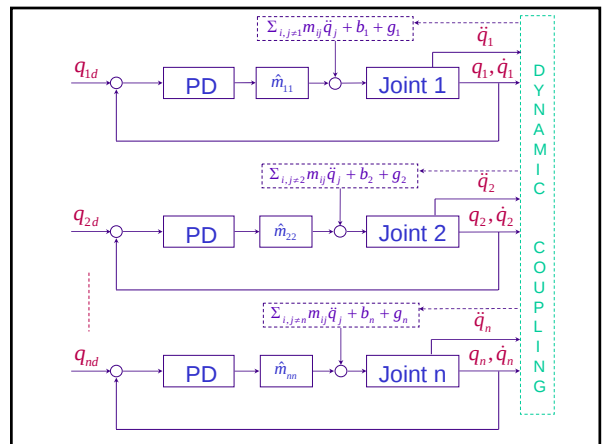
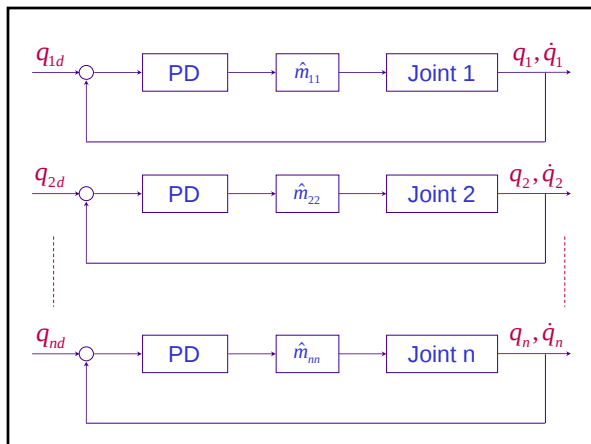
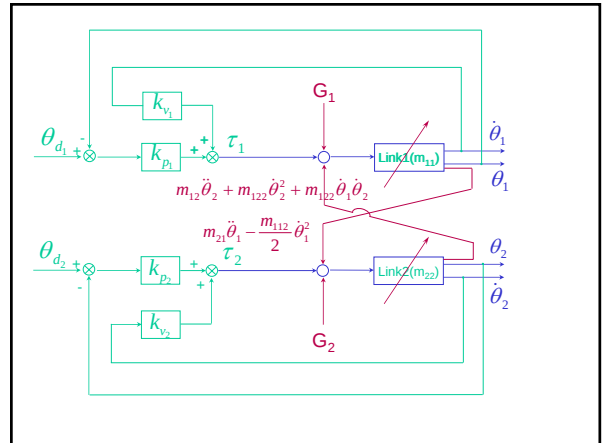


$$M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta) = \tau$$

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} m_{112} \\ 0 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 0 & m_{122} \\ -\frac{m_{112}}{2} & 0 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$$

$$m_{11}\ddot{\theta}_1 + m_{12}\ddot{\theta}_2 + m_{112}\dot{\theta}_1\dot{\theta}_2 + m_{122}\dot{\theta}_2^2 + G_1 = \tau_1$$

$$m_{22}\ddot{\theta}_2 + m_{21}\ddot{\theta}_1 - \frac{m_{112}}{2}\dot{\theta}_1^2 + G_2 = \tau_2$$



PD Control Stability

$$M(q)\ddot{q} + B(q)[\dot{q}\dot{q}] + C(q)[\dot{q}^2] + G(\theta) = \tau$$

$$\tau = -k_p(q - q_d) - k_v\dot{q}$$

$$V_d = 1/2k_p(q - q_d)^2$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} + \frac{\partial V_s}{\partial q} = \tau \frac{\partial V_d}{\partial q} - k_v\dot{q}$$

PD Control Stability

$$M(q)\ddot{q} + B(q)[\dot{q}\dot{q}] + C(q)[\dot{q}^2] + G(\theta) = \tau$$

$$\tau = -k_p(q - q_d) - k_v\dot{q}$$

$$V_d = 1/2k_p(q - q_d)^T(q - q_d)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} + \frac{\partial (V_s - V_d)}{\partial q} = \tau_s$$

$$\tau_s = -k_v\dot{q} \text{ with } \tau_s^T \dot{q} < 0 \text{ for } \dot{q} \neq 0; k_v > 0$$

Performance

High Gains $\rightarrow$ better disturbance rejection

Gains are limited by

structural flexibilities
time delays (actuator-sensing)
sampling rate

$$\omega_n \leq \frac{\omega_{res}}{2} \quad \leftarrow \text{lowest structural flexibility}$$

$$\omega_n \leq \frac{\omega_{delay}}{3} \quad \leftarrow \text{largest delay} \left(\frac{2\pi}{\tau_{delay}} \right)$$

$$\omega_n \leq \frac{\omega_{sampling-rate}}{5}$$

Nonlinear Dynamic Decoupling

$$M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta) = \tau$$

$$\tau = \hat{M}(\theta)\tau' + \hat{V}(\theta, \dot{\theta}) + \hat{G}(\theta)$$

$$1. \ddot{\theta} = (M^{-1}\hat{M})\tau' + M^{-1}[(V - \hat{V}) + (G - \hat{G})]$$

with perfect estimates

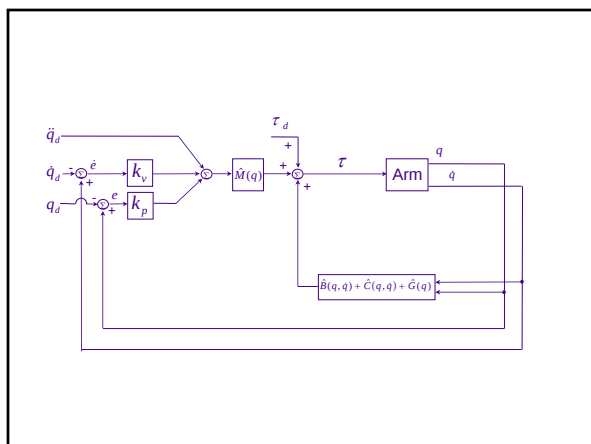
$$1. \ddot{\theta} = \tau' + \varepsilon(t)$$

τ' : input of the unit-mass systems

$$\tau' = \ddot{\theta}_d - k'_v(\dot{\theta} - \dot{\theta}_d) - k'_p(\theta - \theta_d)$$

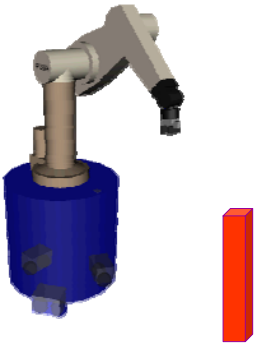
Closed-loop

$$\ddot{E} + k'_v\dot{E} + k'_pE = 0 + \varepsilon(t)$$

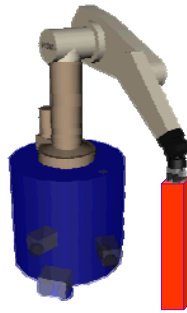


Task Oriented Control

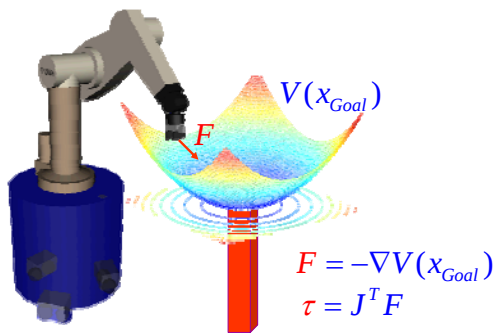
Joint Space Control



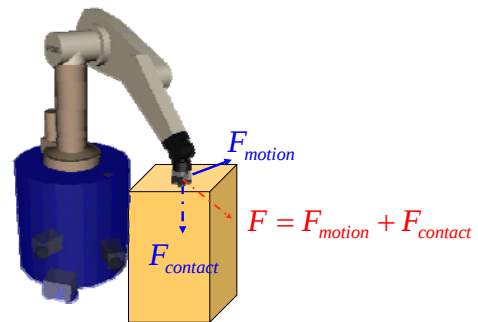
Joint Space Control



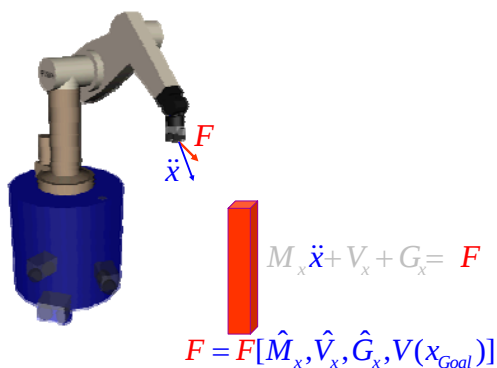
Operational Space Control



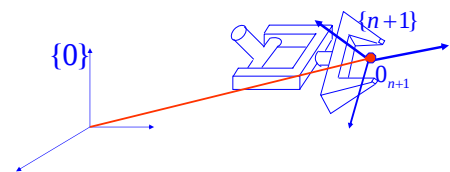
Unified Motion & Force Control



Operational Space Dynamics



Task-Oriented Equations of Motion



Non-Redundant Manipulator ; $n = m$

$$x = (x_1 \ x_2 \ \dots \ x_m)^T$$

$$q = (q_1 \ q_2 \ \dots \ q_n)^T$$

Equations of Motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = F$$

with

$$L(x, \dot{x}) = K(x, \dot{x}) - U(x)$$

$$x = \begin{pmatrix} x \\ y \\ z \\ \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Operational Space Dynamics

$$M_x(x) \ddot{x} + V_x(x, \dot{x}) + G_x(x) = F$$

x : End-Effector Position and Orientation

$M_x(x)$: End-Effector Kinetic Energy Matrix

$V_x(x, \dot{x})$: End-Effector Centrifugal and Coriolis forces

$G_x(x)$: End-Effector Gravity forces

F : End-Effector Generalized forces

Joint Space/Task Space Relationships

Kinetic Energy

$$K_x(x, \dot{x}) \equiv K_q(q, \dot{q})$$

$$\frac{1}{2} \dot{x}^T M_x(x) \dot{x} \equiv \frac{1}{2} \dot{q}^T M(q) \dot{q}$$

Using $\dot{x} = J(q) \dot{q}$

$$\frac{1}{2} \dot{q}^T (J^T M_x J) \dot{q} \equiv \frac{1}{2} \dot{q}^T M \dot{q}$$

Joint Space/Task Space Relationships

$$M_x(x) = J^{-T}(q) M(q) J^{-1}(q)$$

$$V_x(x, \dot{x}) = J^{-T}(q) V(q, \dot{q}) - M_x(q) h(q, \dot{q})$$

$$G_x(x) = J^{-T}(q) G(q)$$

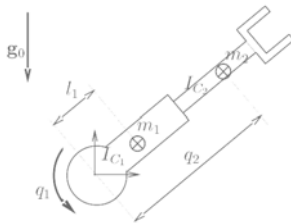
where $h(q, \dot{q}) \doteq \dot{J}(q) \dot{q}$

Example

$$q_2 = d_2$$

$$x = \begin{bmatrix} d_2 c1 \\ d_2 s1 \end{bmatrix}$$

$${}^0 J = \begin{bmatrix} -d_2 s1 & c1 \\ d_2 c1 & s1 \end{bmatrix}$$



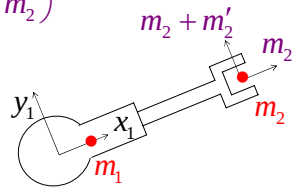
$${}^0 J = \begin{bmatrix} -d_2 s1 & c1 \\ d_2 c1 & s1 \end{bmatrix}$$

$${}^0 J = \begin{pmatrix} c1 & -s1 \\ s1 & c1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ d_2 & 0 \end{pmatrix} \quad \text{where } {}^1 J = \begin{pmatrix} 0 & 1/d_2 \\ 1 & 0 \end{pmatrix}$$

$${}^1 J^{-1} = \begin{pmatrix} 0 & 1/d_2 \\ 1 & 0 \end{pmatrix};$$

$${}^1 M_x = \begin{pmatrix} 0 & 1 \\ 1/d_2 & 0 \end{pmatrix} \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix} \begin{pmatrix} 0 & 1/d_2 \\ 1 & 0 \end{pmatrix}$$

$${}^1M_x = \begin{pmatrix} m_2 & 0 \\ 0 & m_2 + m'_2 \end{pmatrix}$$

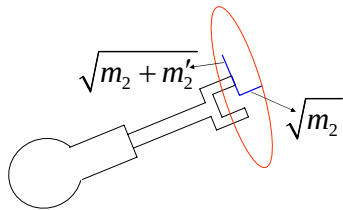


$$m'_2 = \frac{I_{331} + I_{332} + m_1 l_1^2}{d_2^2}$$

$${}^0M_x = \begin{pmatrix} c1 & -s1 \\ s1 & c1 \end{pmatrix} \begin{pmatrix} m_2 & 0 \\ 0 & m_2^+ \end{pmatrix} \begin{pmatrix} c1 & s1 \\ -s1 & c1 \end{pmatrix}$$

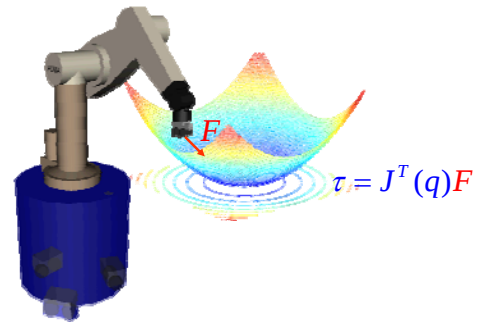
$$m_2^+ = m_2 + m'_2$$

$${}^0M_x = \begin{pmatrix} m_2 + m'_2 s1^2 & -m'_2 s c1 \\ -m'_2 s c1 & m_2 + m'_2 c1^2 \end{pmatrix}$$



$${}^0\Lambda = \begin{pmatrix} m_2 + m'_2 s1^2 & -m'_2 s c1 \\ -m'_2 s c1 & m_2 + m'_2 c1^2 \end{pmatrix}$$

End-Effector Control



Passive Systems (Stability)

$$V_{goal} = \frac{1}{2} k_p (x - x_g)^T (x - x_g)$$

$$\text{System } \frac{d}{dt} \left(\frac{\partial(K-V)}{\partial \dot{x}} \right) - \frac{\partial(K-V)}{\partial x} = F$$

$$\Downarrow F = -\frac{\partial}{\partial x} (V_{goal} - \hat{V})$$

$$\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{x}} \right) - \frac{\partial(K - V_{goal})}{\partial x} = 0 \quad \text{Conservative Forces}$$

Stable

Asymptotic Stability

$$\text{a system } \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{x}} \right) - \frac{\partial(K - V_{goal})}{\partial x} = F_s$$

is asymptotically stable if

$$F_s^T \dot{x} < 0 \quad ; \quad \text{for } \dot{x} \neq 0$$

$$F_s = -k_v \dot{x} \rightarrow k_v > 0$$

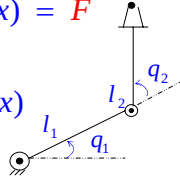
Control

$$F = -k_p (x - x_{goal}) + \hat{G}_x - k_v \dot{x}$$

Example 2-d.o.f arm: Non-Dynamic Control

$$M_x(x)\ddot{x} + V_x(x, \dot{x}) + G_x(x) = F$$

$$F = -k_p(x - x_g) - k_v\dot{x} + \hat{G}(x)$$



$$(m_1^*c^2 + m_2)\ddot{x} + m_1^*\ddot{y} + V_{x1} = -k_p(x - x_g) - k_v\dot{x}$$

$$(m_1^*c^2 + m_2)\ddot{y} + m_1^*\ddot{x} + V_{x2} = -k_p(y - y_g) - k_v\dot{y}$$

Closed loop behavior

$$m_{11}(q)\ddot{x} + k_v\dot{x} + k_p(x - x_g) = -(m_1^*\ddot{y} + V_{x1})$$

$$m_{22}(q)\ddot{y} + k_v\dot{y} + k_p(y - y_g) = -(m_1^*\ddot{x} + V_{x2})$$

Nonlinear Dynamic Decoupling

Model

$$M_x(x)\ddot{x} + V_x(x, \dot{x}) + G_x(x) = F$$

Control Structure

$$F = \hat{M}(x)F' + \hat{V}_x(x, \dot{x}) + \hat{G}_x(x)$$

Decoupled System

$$I\ddot{x} = F'$$

with $\tau = J^T F$

Perfect Estimates

$$I\ddot{x} = F'$$

F' input of decoupled end-effector

Goal Position Control

$$F' = -k_v\dot{x} - k_p(x - x_g)$$

Closed Loop

$$I\ddot{x} + k_v\dot{x} + k_p(x - x_g) = 0$$

Trajectory Tracking

Trajectory: $x_d, \dot{x}_d, \ddot{x}_d$

$$F' = I\ddot{x}_d - k_v(\dot{x} - \dot{x}_d) - k_p(x - x_d)$$

$$(\ddot{x} - \ddot{x}_d) + k_v(\dot{x} - \dot{x}_d) + k_p(x - x_d) = 0$$

or $\ddot{\varepsilon}_x + k_v\dot{\varepsilon}_x + k_p\varepsilon_x = 0$

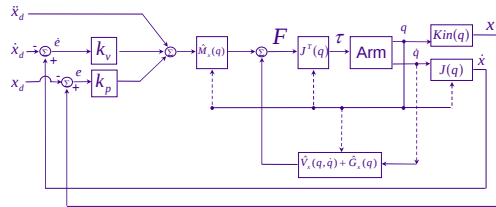
with $\varepsilon_x = x - x_d$

In joint space

$$\ddot{\varepsilon}_q + k_v\dot{\varepsilon}_q + k_p\varepsilon_q = 0$$

with $\varepsilon_q = q - q_d$

Task-Oriented Control

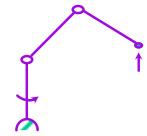


Compliance

$$I \ddot{x} = F'$$

$$F' = - \begin{pmatrix} k'_{p_x} & 0 & 0 \\ 0 & k'_{p_y} & 0 \\ 0 & 0 & k'_{p_z} \end{pmatrix} (x - x_d) - k'_v \dot{x}$$

set to zero



$$\ddot{x} + k'_v \dot{x} + k'_{p_x} (x - x_d) = 0$$

$$\ddot{y} + k'_v \dot{y} + k'_{p_y} (y - y_d) = 0$$

$$\ddot{z} + k'_v \dot{z} = 0$$

Compliance along Z

Stiffness

$$\ddot{z} + k'_v \dot{z} + k'_{p_z} (z - z_d) = 0$$

determines stiffness along z

$$\text{Closed-Loop Stiffness: } \hat{M}_x k'_p = k_p$$

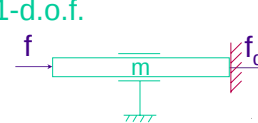
$$F = K_x (x - x_d)$$

$$\tau = J^T F = J^T K_x \Delta x = (J^T K_x J) \Delta \theta = K_\theta \Delta \theta$$

$$K_\theta = J^T(\theta) K_x J(\theta)$$

Force Control

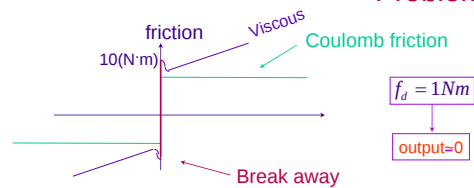
1-d.o.f.



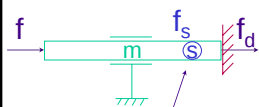
$$m\ddot{x} = f$$

$$\text{set } f = f_d$$

Problem



Force Sensing



$$\otimes \equiv \text{spring} ; f_s = k_s x$$

$$m\ddot{x} + k_s x = f$$

At static Equilibrium

$$f_s = f_d \Rightarrow f = f_d$$

Dynamics

$$m\ddot{x} + k_s x = f_d + f_{\text{Dynamic}}$$

Dynamics

$$m\ddot{x} + k_s x = f$$

$$f_s = k_s x$$

$$\dot{f}_s = k_s \dot{x}$$

$$\frac{m}{k_s} \ddot{f}_s + f_s = f$$

$$\ddot{f}_s = k_s \ddot{x}$$

$$\text{Control} \quad f_d + \frac{m}{k_s} (-k'_{p_f} (f_s - f_d) - k'_{v_f} \dot{f}_s)$$

Closed Loop

$$\frac{m}{k_s} [\ddot{f}_s + k'_{v_f} \dot{f}_s + k'_{p_f} (f_s - f_d)] + f_s = f_d$$

Steady-State error

$$\frac{m}{k_s}(\ddot{f}_s + k'_{v_f} \dot{f}_s + k'_{p_f}(f_s - f_d)) + (f_s - f_d) = 0$$

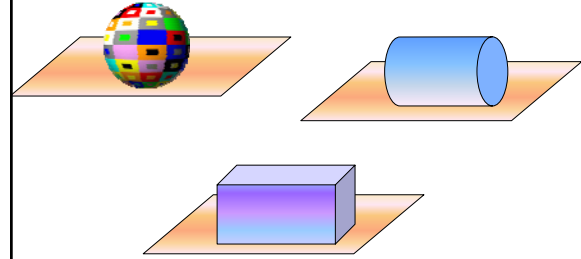
$f_{dist} \uparrow$

$$\ddot{f}_s = \dot{f}_s = 0$$

$$\left(\frac{mk'_{p_f}}{k_s} + 1\right)e_f = f_{dist}$$

$$e_f = \frac{f_{dist}}{1 + \frac{mk'_{p_f}}{k_s}}$$

Task Description

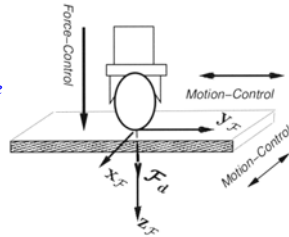


Task Specification

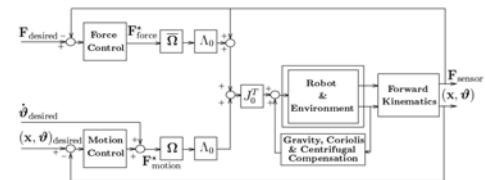
$$F = \Omega F_{motion} + \bar{\Omega} F_{force}$$

Selection matrix

$$\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \bar{\Omega} = I - \Omega$$



Unified Motion & Force Control



Two decoupled Subsystems

$$\Omega \dot{g} = \Omega F_{motion}^*$$

$$\bar{\Omega} \dot{g} = \bar{\Omega} F_{force}^*$$